Taming the Electrostatic Alfvén Wave and Ampere Cancellation Problems

Ilon Joseph

Fusion Energy Sciences Program
Lawrence Livermore National Laboratory

Transport Task Force Workshop
San Diego, CA
May 9th, 2018

*Prepared under the auspices of the U.S. DOE by LLNL under Contract DE-AC52-07NA27344.
LLNL-POST-750926

Abstract (1/2)

Two well-known issues arise when attempting to solve equations for magnetized plasmas that are accurate for both large scale magnetohydrodynamics (MHD) and for small scale microturbulence: the “electrostatic Alfven wave problem” and the “Ampere cancellation problem.” These issues must be solved for kinetic simulations in order to treat electron physics both accurately and efficiently. For example, recent ELM simulations [1] achieved longer time steps by eliminating the electrostatic Alfven wave in a physically inspired, but ad hoc manner. In this work, it is demonstrated that these two issues are related and can be addressed by using the generalized Ohm’s law, by including both electron inertia and the inductive electric field.

The “electrostatic Alfven wave problem” arises when one assumes that the scales of interest are short wavelength. This assumption typically allows one to take the electrostatic limit $E = -\nabla\Phi$ and neglect the inductive electric field. However, the inductive electric field $-\partial_t A$ must be retained whenever the perpendicular scale lengths are larger than the plasma skin depth. The issue is that the skin depth is small and is typically similar in magnitude to the gyroradius. Hence, the electrostatic assumption is invalid for meso- and macro-scale phenomena, such as MHD. The penalty for this incorrect assumption is an incorrect dispersion relation for the shear Alfvén wave that has a frequency that grows without bound at long perpendicular wavelengths. When using an explicit time integration strategy, this would impose a very stringent Courant-Freidrichs-Lewy (CFL) criterion.

References

E. Shi, A. H. Hakim, and G. W. Hammett, Phys. Plasmas 22, 022504 (2015).

Abstract (2/2)

The “Ampere cancellation problem” appears when one attempts to use Ampere’s law rather than Ohm’s law in order to determine the magnetic field. For spatial scales shorter than the plasma skin depth, the magnetostatic assumption is valid and Ampere’s law can be used. However, for longer spatial scales, the plasma is almost an ideal conductor. This yields a strong constraint that the electric field must nearly cancel the sum of all of the other forces that the electrons experience. Hence the name “cancellation problem.” Typically, the cancellation occurs between the parallel electric field and the parallel electron pressure gradient. A kinetic simulation that uses the generalized Ohm’s law to determine the inductive electric field naturally provides the cancellation required to solve these problems. Solving Ohm’s law provides an update for the vector potential that interpolates between the short and long wavelength limits. Inserting this electric field into the kinetic equation provides the needed cancelation between forces at long wavelength.

Outline

- Motivation: Electrostatic Alfvén Wave Problem
- Quasineutraity & “Poisson Cancellation” Problem
- Electrostatic Shear Alfvén Wave Problem
- Ampere Cancellation Problem
- Conclusion

Electrostatic Shear Alfvén Wave (ES-AW) Problem

- **ES-AW can be derived from the vorticity & electron force balance eqs.**

$$\partial_t \vec{\nabla} \cdot \frac{m_i n_i}{B^2} \vec{\nabla}_\perp \phi = \vec{\nabla} \cdot J_\| \hat{b}$$

$$\partial_t J_\| = -e^2 n_e E_\| / m_e = e^2 n_e \nabla_\| \phi / m_e$$

- **ES-AW dispersion relation is singular for** $k_\perp / k_\| \to 0$

$$\omega = k_\| V_{Te} / k_\perp \rho_i = k_\| V_a / k_\perp \lambda_p .$$

- **If we include the inductive electric field ...**

$$\nabla_\perp^2 A_\| = \mu_0 J_\|$$

$$\partial_t J_\| - e^2 n_e \partial_t A_\| / m_e = e^2 n_e \nabla_\| \phi / m_e$$

- **... then the skin depth** $\lambda_p = c/\omega_p$ **controls the limit**

$$\omega = k_\| V_a / \left[1 + (k_\perp \lambda_p)^2 \right]^{1/2}$$

- **Conclusion: the electrostatic assumption is invalid in the long wavelength** $k_\perp \lambda_p < 1$ **regime and** $\partial_t A_\|$ **must be retained**

Quasineutrality Constraint

- **Quasineutrality is the assumption that the charge density is much smaller than the particle density, and hence, that**

$$\partial_t \rho = -\vec{\nabla} \cdot \vec{J} \ll |\nabla_i J_j|$$

- **The quasineutrality (QN) constraint is**

$$\partial_t \rho = -\vec{\nabla} \cdot \vec{J} = 0 \qquad \partial_t \vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot \partial_t \vec{J} = 0$$

- **QN2 implies the charge/mass-weighted sum of the forces vanishes**

$$\vec{\nabla} \cdot \partial_t \vec{J} = -\sum_j e_j \vec{F}_j / m_j = 0 \qquad \boxed{\vec{\nabla} \cdot \epsilon_0 \Omega_p^2 \vec{E} = -\sum_j \vec{\nabla} \cdot e_j \vec{F}_j' / m_j}$$

  - Plasma frequency $\Omega_{pj}^2 = e_j^2 n_j / \epsilon_0 m_j$ and total plasma frequency $\Omega_p^2 = \sum_j \Omega_{pj}^2$
  - Force density $\vec{F}_j = e_j n_j \vec{E} - \nabla \cdot P_j + \ldots$
  - Force density without electric force $\vec{F}_j' = \vec{F}_j - e_j n_j \vec{E}$

Maxwell's Equations and the Generalized Ohm's Law

- **Maxwell's Eqs.**

$$\partial_t \vec{B} = -\vec{\nabla} \times \vec{E} \qquad \partial_t \epsilon_0 \vec{E} = \vec{\nabla} \times \vec{B}/\mu_0 - \vec{J}$$

- **Combining Maxwell's Eqs. leads to the generalized Ohm's law**

$$\partial_t^2 \epsilon_0 \vec{E} + \vec{\nabla} \times \vec{\nabla} \times \vec{E}/\mu_0 = -\partial_t \vec{J} = -\sum_j e_j n_j \vec{F}_j / m_j$$

- **The full equation that determines $\vec{E}$ is**

$$\partial_t^2 \epsilon_0 \vec{E} + \vec{\nabla} \times \vec{\nabla} \times \vec{E}/\mu_0 + \omega_p^2 \epsilon_0 \vec{E} + \sum_j \vec{J}_j \times \vec{\Omega}_{cj} = -\sum_j e_j n_j \vec{f}_j / m_j$$

  - Larmor frequency $\Omega_{cj} = e_j \vec{B}/m_j$
  - Force density $\vec{F}_j = e_j n_j \vec{E} - \nabla \cdot P_j + \ldots$
  - Force density without Lorentz force $\vec{f}_j = \vec{F}_j - e_j n_j (\vec{E} + \vec{v}_j \times \vec{B})$

Enforcing Quasineutrality for Unmagnetized Ohm's Law

- **Eliminate light wave and plasma wave by assuming $\omega/kc \ll 1$**

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}/\mu_0 + \omega_p^2 \epsilon_0 \vec{E} \simeq \sum_j e_j \nabla \cdot P_j / m_j$$

$$\vec{E} = \frac{1}{\omega_p^2 (1 + k_\perp^2 \lambda_p^2)} \sum_j \frac{e_j \nabla \cdot P_j}{\epsilon_0 m_j} \simeq \frac{-1}{(1 + k_\perp^2 \lambda_p^2)} \frac{\nabla \cdot P_e}{e n_e}$$

- **Insert into electron kinetic equation**

$$\partial_t f_e + \vec{\nabla}_x \cdot \vec{v} f_e - \vec{\nabla}_v \cdot e\vec{E} f_e / m_e = 0$$

$$\partial_t f_e + \vec{\nabla}_x \cdot \vec{v} f_e + \vec{\nabla}_v \cdot (\nabla \cdot P_e) f_e / m_e n_e (1 + k_\perp^2 \lambda_p^2) = 0$$

- **Electron continuity & momentum = Generalized Ohm's law**

$$\partial_t n_e + \nabla \cdot n_e v_{e\|} \hat{b} = 0 \qquad \partial_t m_e n_e \vec{v}_e + \frac{k_\perp^2 \lambda_p^2}{1 + k_\perp^2 \lambda_p^2} (\nabla \cdot P_e) = 0.$$

- **Disp. rel. for electron density waves now requires $k_\| k_\perp \neq 0$**

$$\omega^2 = k_\|^2 {V_{Te}}^2 \times k_\perp^2 \lambda_p^2 / \left[1 + k_\perp^2 \lambda_p^2\right]$$

Magnetized Parallel Ohm's Law

- **Parallel Ohm's law**

$$\vec{\nabla}_\perp^2 E_\| / \mu_0 + \omega_p^2 \epsilon_0 E_\| \simeq \sum_j e_j \nabla_\| P_j / m_j$$

$$E_\| = \frac{1}{\omega_p^2 (1 + k_\perp^2 \lambda_p^2)} \sum_j \frac{e_j (\nabla \cdot P_j)_\|}{\epsilon_0 m_j} \simeq \frac{-1}{(1 + k_\perp^2 \lambda_p^2)} \frac{(\nabla \cdot P_e)_\|}{e n_e}$$

- **Insert into electron gyro-kinetic equation** $F = f B_\|^*$

$$\partial_t F_e + \vec{\nabla}_x \cdot (v_\| \hat{b} + \vec{v}_d) F_e - \partial_{v_\|} e E_\| F_e / m_e = 0$$

$$\partial_t F_e + \vec{\nabla}_x \cdot (v_\| \hat{b} + \vec{v}_d) F_e + \partial_{v_\|} (\nabla \cdot P_e)_\| F_e / m_e n_e (1 + k_\perp^2 \lambda_p^2) = 0$$

- **Electron continuity & momentum = Generalized Ohm's law**

$$\partial_t n_e + \nabla \cdot n_e v_{e\|} \hat{b} = 0 \qquad \partial_t m_e n_e v_{e\|} + \frac{k_\perp^2 \lambda_p^2}{1 + k_\perp^2 \lambda_p^2} (\nabla \cdot P_e)_\| = 0.$$

- **Disp. rel. for electron density waves similar to unmagnetized case**

$$\omega^2 = k_\|^2 {V_{Te}}^2 \times k_\perp^2 \lambda_p^2 / \left[1 + k_\perp^2 \lambda_p^2\right].$$

Magnetized Perpendicular Ohm's Law $\rightarrow$ Vorticity Eq.

- $\vec{E}_\perp$ **is still electrostatic** $\partial_t \vec{A}_\perp \ll \nabla_\perp \phi$
- **Use total force balance to determine perpendicular current**

$$\partial_t \left(\vec{B} \times \frac{\mu_0}{V_a^2} \vec{\nabla}_\perp \phi \right) + \nabla \cdot P = \vec{J} \times \vec{B}.$$

  - Aflvén speed $V_a^{-2} = \sum_j \mu_0 m_j n_j / B^2$
  - Total pressure $P = \sum_j P_j$

- **Now quasineutrality** $\vec{\nabla} \cdot \vec{J} = 0 \rightarrow$ **Vorticity Eq.**

$$\vec{J} = J_\parallel \hat{b} + \frac{\hat{b}}{B} \times \left(\nabla P - \partial_t \vec{B} \times \frac{\mu_0}{V_a^2} \vec{\nabla}_\perp \phi \right)$$

$$\partial_t \vec{\nabla}_\perp \cdot \frac{\mu_0}{V_a^2} \vec{\nabla}_\perp \phi = \vec{\nabla} \cdot J_\parallel \hat{b} + \vec{\nabla} \cdot \frac{\hat{b}}{B^2} \times \nabla \cdot P$$

- **Combining with parallel Ohm's law leads to correct shear Alfvén wave dispersion relation**

$$\left[1 + k_\perp^2 \lambda_p^2\right] \partial_t A_\parallel = -\nabla_\parallel \phi + \dots$$

$$\omega^2 = k_\parallel^2 V_a^2 / \left[1 + k_\perp^2 \lambda_p^2\right]$$

Conclusions

- **Electrostatic Alfvén wave problem can be solved by including inductive electric field $\partial_t \vec{A}$**
  - Electrostatic assumption invalid for wavelengths longer than the skin depth $k_\perp \lambda_p < 1$
- **Ampere cancellation problem can be solved by using generalized Ohm's law instead of Ampere's law**
  - **Solution:** Solve generalized Ohm's law for $\vec{E}$ and insert into the kinetic equation
- **Still, one must ensure numerically accurate "cancellation" b/w electric field & electron pressure**

$$(\nabla \cdot P_e)_\parallel + e n_e E_\parallel \simeq \frac{\lambda_p^2 k_\perp^2}{(1 + \lambda_p^2 k_\perp^2)} (\nabla \cdot P_e)_\parallel$$